A proof concerning Swiss cheese sets.

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Introduction

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(a) A **Swiss cheese set** X is a compact plane set produced by deleting from the complex plane the elements of a collection $\tilde{D} := \mathcal{D} \cup \{\mathbb{C} \backslash \Delta\}$, where $\mathcal{D}$ is a set of open discs in the complex plane and Δ is a closed disc.

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(b) A **Swiss cheese** $d : S \to D$ is a map from a subset $S \subseteq \mathbb{N} \cup \{0\}$ into D such that $d(S)$ is a collection that defines a Swiss cheese set where $d(S \backslash \{0\})$ is a set of open discs and $d(0)$ is the complement of a closed disc.

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With these definitions to hand we now look at Heath's theorem on Swiss cheese sets which is the central issue for us.

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Heath's theorem on Swiss cheese sets.

Theorem

For every Swiss cheese $h \in H$ there is a classical Swiss cheese $h' \in H$ with $X_{h'} \subseteq X_h$.

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The existing Zorn's lemma proof by Heath and Feinstein is elegant. However the proof presented here is perhaps more intuitive giving a nice example of the application of transfinite induction and the use of cardinality in proof.

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(1) If $h \in H$ is a classical Swiss cheese then define $f(h) := h$.

(2) If $h \in H$ is not classical then for $h : S \to D$ let

$$I := \{(n, m) \in S^2 : \bar{h}(n) \cap \bar{h}(m) \neq \emptyset, n \neq m\}.$$

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$$I := \{(n, m) \in S^2 : \bar{h}(n) \cap \bar{h}(m) \neq \emptyset, n \neq m\}.$$

We then have lexicographic ordering on I given by

$$(n, m) \lesssim (n', m') \text{ iff } n < n' \text{ or } (n = n' \text{ and } m \leq m').$$

Since this is a well-ordering on I, let (n, m) be the minimum element of I and hence note that $m \neq 0$.

Proof. The map $f : H \to H$.

(2) From the last slide, (n, m) is the minimum in I. Since $h \in H$ we have $\sum_{k \in S \setminus \{0\}} r(h(k)) = r(\mathbb{C} \setminus h(0)) - \delta$ for some $\delta > 0$.

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Now, by a lemma of Heath, there exists $E \in D$ with $h(n) \cup h(m) \subseteq E$ such that for $f(h) : S \setminus \{m\} \to D$,

$$f(h)(k) := \begin{cases} h(k) & \text{for } k \neq n \\ E & \text{for } k = n \end{cases}, k \in S \setminus \{m\}$$

we have $\sum_{k \in S \setminus \{0, m\}} r(f(h)(k)) \leq r(\mathbb{C} \setminus f(h)(0)) - \delta$ and so $f(h) \in H$.

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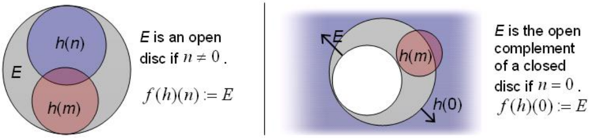
Figure 1: The suitable open set $E \in D$ satisfying Heath's lemma.



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(b) Further, for the Swiss cheese sets given by $X_h := \mathbb{C} \setminus \cup_{n \in S} h(n)$ we have $X_{f(h)} \subseteq X_h$ noting that for $n \in S'$ we have $h(n) \subseteq f(h)(n)$.

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Our aim is to prove that for every $h \in H$ there is a classical Swiss cheese $h' \in H$ with $X_{h'} \subseteq X_h$.

To this end we will use $f : H \to H$ to construct an ordinal sequence of Swiss cheeses and then apply a cardinality argument to show that this ordinal sequence must stabilise at a classical Swiss cheese.

Proof. A sequence through $f : H \to H$.

Notation

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So by the properties of $f : H \to H$ we obtain a nested sequence of Swiss cheese sets:

$X_h \supseteq X_{h^1} \supseteq X_{h^2} \supseteq \cdots$

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and a corresponding nested sequence of subsets of $\mathbb{N} \cup \{0\}$, each including 0:

$S \supseteq S_1 \supseteq S_2 \supseteq \cdots$ where $S_\alpha = S_{\alpha+1}$ iff h^α is classical.

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We need to extend these sequences so that they become ordinal sequences that preserve the existing properties.

Proof. An ordinal sequence.

Extending the sequence to all ordinals.

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(a) If α is a successor ordinal then define $h^{\alpha} := f(h^{\alpha-1})$.

(b) If α is a limit ordinal then define $h^{\alpha} : S_{\alpha} \to D$ by,

$$n \mapsto \cup_{\beta<\alpha} h^{\beta}(n) \text{ with } S_{\alpha} := \cap_{\beta<\alpha} S_{\beta}.$$

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We need to show that h^α is in H and that it has the required properties. We use transfinite induction.

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For α an ordinal number let $P(\alpha)$ be the proposition that:

(α,1) $h^\alpha \in H$, i.e.

- **(α,1.1)** $0 \in S_\alpha$,
- **(α,1.2)** $h^\alpha(0)$ is the complement of a closed disc and $h^\alpha(n)$ is an open disc for $n \in S_\alpha \setminus \{0\}$,
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Inductive step: for α a **successor ordinal**, $P(\alpha)$ is immediate.

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We have $S_\alpha := \cap_{\beta<\alpha} S_\beta$ giving, $\forall \beta \leq \alpha$, $S_\alpha \subseteq S_\beta$. Hence ($\alpha$,2) holds.

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Also $\forall \beta < \alpha$ we have $0 \in S_\beta$ by (β,1.1). So $0 \in S_\alpha$ i.e. (α,1.1) holds.

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We now show (α,1.2) as follows.

(1*) For $n \in S_\alpha \setminus \{0\}$, $\{h^\beta(n) : \beta < \alpha\}$ is a nested increasing family of open discs by (β,1.2) and (β,4).

(2*) Further, $\{\mathbb{C} \setminus h^\beta(0) : \beta < \alpha\}$ is a nested decreasing family of closed discs by (β,1.2) and (β,4).

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To proceed we use two lemmas by Heath.

Proof. An ordinal sequence.

Lemma 1, (Heath)

Lemma

Let $\mathcal{F}$ be a non-empty, nested collection of open discs in $\mathbb{C}$, such that $\sup\{r(E) : E \in \mathcal{F}\} < \infty$*. Then $\cup\mathcal{F}$ is an open disc D. Further, for $\mathcal{F}$ ordered by inclusion,* $r(D) = \lim_{E\in\mathcal{F}} r(E)$.

Proof. An ordinal sequence.

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Lemma 2, (Heath)

Lemma

Let $\mathcal{F}$ be a non-empty, nested collection of closed discs in $\mathbb{C}$, such that $\inf\{r(E) : E \in \mathcal{F}\} > 0$. Then $\cap\mathcal{F}$ is a closed disc Δ. Further, for $\mathcal{F}$ ordered by reverse inclusion, $r(\Delta) = \lim_{E\in\mathcal{F}} r(E)$.

Proof. An ordinal sequence.

Now for $n \in S_\alpha \backslash \{0\}$ and $\beta < \alpha$ we have
$r(h^\beta(n)) \leq \sum_{m \in S_\beta \backslash \{0\}} r(h^\beta(m)) \leq r(\mathbb{C} \backslash h^\beta(0)) - \delta_{h^\beta} \leq r(\mathbb{C} \backslash h(0)) - \delta_h$,
by (β,1.3) and (2*).

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by (β,1.3) and (2*).

Hence $\sup\{r(h^\beta(n)) : \beta < \alpha\} \leq r(\mathbb{C} \backslash h(0)) - \delta_h$.
So by (1*) and lemma 1 we have for $n \in S_\alpha \backslash \{0\}$ that

$$h^\alpha(n) := \cup_{\beta < \alpha} h^\beta(n)$$

is an open disc with,

$$r(h^\alpha(n)) = \lim_{\beta < \alpha} r(h^\beta(n)) \leq r(\mathbb{C} \backslash h(0)) - \delta_h.$$

Proof. An ordinal sequence.

Now for $\beta < \alpha$ we have $r(\mathbb{C} \backslash h^{\beta}(0)) \geq \delta_{h^{\beta}} \geq \delta_{h}$ by (β,1.3).

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Now for $\beta < \alpha$ we have $r(\mathbb{C} \setminus h^\beta(0)) \geq \delta_{h^\beta} \geq \delta_h$ by (β,1.3).

Hence $\inf\{r(\mathbb{C} \setminus h^\beta(0)) : \beta < \alpha\} \geq \delta_h$.
So by De Morgan, (2*) and lemma 2 we have

$$\mathbb{C} \setminus h^\alpha(0) := \mathbb{C} \setminus \cup_{\beta<\alpha} h^\beta(0) = \cap_{\beta<\alpha} \mathbb{C} \setminus h^\beta(0)$$

is a closed disc with,

$$r(\mathbb{C} \setminus h^\alpha(0)) = \lim_{\beta<\alpha} r(\mathbb{C} \setminus h^\beta(0)) \geq \delta_h.$$

Hence $h^\alpha(0)$ is the complement of a closed disc and so (α,1.2) holds.

Proof. An ordinal sequence.

We now show that (α,4) holds.
By (β,4) we have $\forall n \in S_\alpha$, $\{h^\beta(n) : \beta < \alpha\}$ is a nested increasing family.
We also have $h^\alpha(n) := \cup_{\beta<\alpha} h^\beta(n)$ so $\forall \beta \leq \alpha$, $h^\beta(n) \subseteq h^\alpha(n)$. Hence ($\alpha$,4) holds.

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We have shown that (α,1.1), (α,1.2), (α,2) and (α,4) all hold. For brevity, (α,1.3) and (α,3) also follow from the inductive hypothesis.

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By (β,4) we have $\forall n \in S_\alpha$, $\{h^\beta(n) : \beta < \alpha\}$ is a nested increasing family.
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We have shown that (α,1.1), (α,1.2), (α,2) and (α,4) all hold. For brevity, (α,1.3) and (α,3) also follow from the inductive hypothesis.

Hence $P(\alpha)$ holds and so by the principle of transfinite induction $P(\alpha)$ holds for all ordinal numbers α.

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Closing argument using cardinality.
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Lemma

For the Swiss cheese h^β we have,
*h^β is classical **iff** (S_α) has stabalised at β, i.e. $S_{\beta+1} = S_\beta$.*

Proof of lemma, follows directly from property (a) of the map $f : H \to H$ and $(\beta, 1.3)$.

Proof. Closing argument.

Toward a contradiction suppose for the first uncountable ordinal ω_1 that $\forall \beta < \omega_1$, (S_α) has not stabalised at β.
Then for each $\beta < \omega_1$ there exists some $n_{\beta+1} \in \mathbb{N}$ such that $n_{\beta+1} \in S^c_{\beta+1}$ but $n_{\beta+1} \notin S^c_\alpha \; \forall \, \alpha \leq \beta$.

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Therefore there exists $\beta < \omega_1$ such that (S_α) has stabalised at β and so by the last lemma h^β is classical.

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In particular, Heath's theorem on Swiss cheese sets has been proven.
□