Generalising Uniform Algebras Over Complete Valued Fields

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We begin with the following definition.

Definition 1.0

Let F be a complete valued field.
Let A be a commutative unital Banach F-algebra.
We say that A has **finite basic dimension** if there exists a finite extension L of F extending F as a valued field such that:

(i) for each proper closed prime ideal J of A, that is the kernel of a bounded multiplicative seminorm on A, $\mathrm{Frac}(A/J)$ is F-isomorphic to a subfield of L;

(ii) there is $g \in \mathrm{Gal}(L/F)$ with $L^g = F$, where $L^g := \{x \in L : g(x) = x\}$.

Representation of uniform algebras, overview.

Let F be $\mathbb{C}$

let A be a commutative unital Banach F-algebra with $\|a^2\| = \|a\|^2$ for all $a \in A$ and finite basic dimension.

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Let F be $\mathbb{C}$ or $\mathbb{R}$

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Representation of uniform algebras, overview.

Let F be $\mathbb{C}$ or $\mathbb{R}$ or $\mathbb{K}$, a locally compact complete nonarchimedean field, let A be a commutative unital Banach F-algebra with $\|a^2\| = \|a\|^2$ for all $a \in A$ and finite basic dimension.

- A is a complex uniform algebra on a compact Hausdorff space,
- or A is a real function algebra on a compact Hausdorff space,
- or A is a nonarchimedean analog of the real function algebras on a Stone space.

Note, here a Stone space is a totally disconnected compact Hausdorff space.

Definition 1.1

Let F and L be complete valued fields such that L is an extension of F as a valued field. Let X be a compact Hausdorff space and let $C_L(X)$ be the Banach algebra of all continuous L-valued functions on X with pointwise operations and the sup norm. If a subset A of $C_L(X)$ satisfies:

(i) A is closed under pointwise operations;

(ii) A is complete with respect to $\|\cdot\|_\infty$;

(iii) $F \subset A$;

(iv) A separates the points of X,

then we will call A an ${}^{L}/_{F}$ **uniform algebra** or just a **uniform algebra** when convenient.

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In the language of Definition 1.1, an ${}^{L}/_{F}$ uniform algebra is a Banach F-algebra of L-valued functions.

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Definition 1.2 (J. Mason 2009)

Let F and L be complete valued fields such that L is a finite extension of F as a valued field. Let X be a compact Hausdorff space and totally disconnected if F is nonarchimedean. Define,

$$C(X,\tau,g) := \{f \in C_L(X) : f(\tau(x)) = g(f(x)) \text{ for all } x \in X\}$$

where: (i) $g \in \mathrm{Gal}({}^{L}/_{F})$;

(ii) $\tau : X \to X$ with $\mathrm{ord}(\tau)|\mathrm{ord}(g)$;

(iii) g and τ are continuous.

We will call $C(X,\tau,g)$ the **basic** ${}^{L}/_{L^g}$ **function algebra on** (X,τ,g), where $L^g := \{x \in L : g(x) = x\}$, or just a **basic function algebra** when convenient.

Definition 1.3 (J. Mason 2009)

Let F and L be complete valued fields such that L is a finite extension of F as a valued field. Let (X, τ, g) conform to the conditions of Definition 1.2 and let A be a subset of the basic ${}^{L}/_{L^g}$ function algebra on (X, τ, g).

If A is also an ${}^{L}/_{L^g}$ uniform algebra then we will call A an ${}^{L}/_{L^g}$ **function algebra on** (X, τ, g).

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With respect to the above definitions the basic ${}^{L}/_{L^g}$ function algebra on (X, τ, g) is always an ${}^{L}/_{L^g}$ uniform algebra.

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Note, in fact $\mathrm{ord}(\tau)|\mathrm{ord}(g)$ is an optimum condition in Definition 1.2 since if we do not include it in Definition 1.2 then $C(X, \tau, g)$ separates the points of X if and only if $\mathrm{ord}(\tau)|\mathrm{ord}(g)$.

Archimedean examples.

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(Ex1) Let $F = \mathbb{R}$, $L = \mathbb{C}$ and X be a compact Hausdorff space.
We have $\mathrm{Gal}({}^{\mathbb{C}}/_{\mathbb{R}}) = \{\mathrm{id}, \bar{z}\}$.

Setting $g = \mathrm{id}$ forces τ to be the identity on X. In this case $C(X, \tau, g) = C_{\mathbb{C}}(X)$ and each ${}^{L}/_{L^g}$ function algebra on (X, τ, g) is a complex uniform algebra.

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On the other hand, setting $g = \bar{z}$ forces τ to be a topological involution on X. In this case the ${}^{L}/_{L^g}$ function algebras on (X, τ, g) are precisely the real function algebras of Kulkarni and Limaye.

Nonarchimedean examples.

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(Ex2) Let $F = \mathbb{Q}_5, L = \mathbb{Q}_5(\sqrt{2})$ with the unique extension of the 5-adic valuation and $X := \{x \in L : |x|_L \leqslant 1\}$.

Let g be the Galois automorphism that sends $\sqrt{2}$ to $-\sqrt{2}$. Here g is an isometry on L and so we can take $\tau = g$. In this case $C(X, \tau, g)$ has the property that every power series in $C(X, \tau, g)$ has $\mathbb{Q}_5$ valued coefficients. However, since $X \subset \mathbb{Q}_5(\sqrt{2})$ these power series are $\mathbb{Q}_5(\sqrt{2})$ valued functions.

(Ex3) Let F, L, X and g be as in Ex2.
We can obtain a function $\omega : L \to \mathbb{Z} \cup \{+\infty\}$ such that for all $x \in L$ we have $|x|_L = 5^{-\omega(x)}$.
Define $\tau(0) = 0$ and for $x \in X \setminus \{0\}$,

$$\tau(x) := \begin{cases} 5x & \text{if} \quad 2 \mid \omega(x) \\ 5^{-1}x & \text{if} \quad 2 \nmid \omega(x). \end{cases}$$

In this case the only power series in $C(X, \tau, g)$ are constants belonging to $\mathbb{Q}_5$.

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However there are elements of $C(X, \tau, g)$ that when restricted to a circle in X about the origin, can be expressed as a power series on the circle.

Before introducing the next theorem we recall the definition below.

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Theorem 1.5 (J. Mason 2010)

Let F be a locally compact complete nonarchimedean valued field with nontrivial valuation.
Let A be a commutative unital Banach F-algebra with $\|a^2\| = \|a\|^2$ for all $a \in A$ and finite basic dimension.
Then:

(i) for some finite extension L of F extending F as a valued field, a character space $\mathcal{M}(A)$ of L valued, multiplicative F-linear functionals can be defined;

(ii) the space $\mathcal{M}(A)$ is a totally disconnected compact Hausdorff space;

(iii) A is isometrically F-isomorphic to a L/F function algebra on $(\mathcal{M}(A), g, g)$ for some $g \in$Gal(L/F).

Residue algebra theorem.

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Theorem 1.6 (J. Mason 2010)

Let F be a locally compact complete nonarchimedean valued field of characteristic zero with nontrivial valuation.
Let L be a finite unramified extension of F with $L^g = F$ for some $g \in \mathrm{Gal}(L/F)$ and let $C(X, \tau, g)$ be a basic L/F function algebra.
Then:

(i) $\mathcal{O} := \{f \in C(X, \tau, g) : \|f\|_\infty \leq 1\}$ is a ring;

(ii) $\mathcal{J} := \{f \in C(X, \tau, g) : \|f\|_\infty < 1\}$ is an ideal of $\mathcal{O}$;

(iii) $\mathcal{O}/\mathcal{J} \cong C(X, \tau, \bar{g})$ where $C(X, \tau, \bar{g})$ is the basic $\bar{L}/\bar{F}$ function algebra on $(X, \tau, \bar{g})$. Here $\bar{F}$ and $\bar{L}$ are respectively the residue fields of F and L whilst $\bar{g}$ is the residue automorphism on $\bar{L}$ induced by g.

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Let X be a compact Hausdorff space, τ a topological involution on X and A a ${}^{\mathbb{C}}/_{\mathbb{R}}$ function algebra on $(X, \tau, \bar{z})$. If $\mathrm{Re}A$ is a ring then $A = C(X, \tau, \bar{z})$.

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Can this be generalised further for $C(X, \tau, g)$?

(Q2) For $f \in C_L(X)$ define $\sigma(f) := g^{\mathrm{ord}(g)-1} \circ f \circ \tau$.
We have $f \in C(X, \tau, g)$ if and only if $\sigma(f) = f$.
Does every higher order algebraic involution on $C_L(X)$ has the form σ for some g and τ?
Aside, for g an isometry on L we automatically have that σ is an isometry on $C_L(X)$.

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There are many open questions.

References:

(1) V. G. Berkovich, Spectral theory and analytic geometry over nonarchimedean fields, Mathematical surveys and monographs, no. 33, American Mathematical Society, 1990.

(2) S. H. Kulkarni and B. V. Limaye, Real function algebras, Monographs and textbooks in pure and applied mathematics, no. 168, Marcel Dekker inc, 1992.

(3) W. H. Schikhof, Ultrametric calculus an introduction to p-adic analysis, Cambridge University Press, 2006.

We will end here.

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More generaly, a metric space (X, d) is called an **ultrametric space** if the metric d satisfies the strong triangle inequality, $d(x, z) \leq \max\{d(x, y), d(y, z)\}$.

For each prime $p \in \mathbb{N}$ there is a nonarchimedean absolute value $|\cdot|_p$ on the field of rational numbers $\mathbb{Q}$. The completion of $\mathbb{Q}$ obtained using $|\cdot|_p$ is the nonarchimedean field $\mathbb{Q}_p$ of p-adic numbers.

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Each $x \in \mathbb{Q}_p^\times$ has a unique p-power series expansion of the form

$$x = \sum_{i \leq n}^{\infty} a_n p^n, \quad a_n \in \{0, \cdots, p-1\}, \quad a_i \neq 0, \quad i \in \mathbb{Z}.$$

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$\mathbb{Q}_p$ is locally compact where as $\mathbb{C}_p$ is not. Further $\mathbb{C}_p$ and $\mathbb{C}$ are isomorphic, $\mathbb{C}_p \cong \mathbb{C}$ as fields.

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Then $x \sim y :\Leftrightarrow |x-y|_p \leqslant r$ is an equivalence relation on $\mathbb{K}$ by the strong triangle inequality. To show transitivity, let $x \sim y$ and $y \sim z$ then,

$$|x-z|_p = |x-y+y-z|_p \leq \max\{|x-y|_p, |y-z|_p\} \leqslant r \quad \text{so} \quad x \sim z.$$

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(b) Algebraically, $\mathbb{K}/_\sim := \{B_r(x) : x \in \mathbb{K}\}$ is an Abelian group.

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(c) Since $\mathbb{K}$ is a disjoint union of $\sim$ equivalence classes, $\mathbb{K}$ is a disjoint union of balls of radius r.

(d) It follows easily that for $y \notin B_r(x)$ we have $B_r(y) \bigcap B_r(x) = \emptyset$. Hence, also noting (a), every $\mathbb{K}$ ball is clopen.

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(g) $\mathbb{K}$ is totally disconnected.
To see this note that for all $r > 0$ and for all $x \in \mathbb{K}$,
$\mathbb{K} = (\mathbb{K} \backslash B_r(x)) \dot{\bigcup} B_r(x)$ is a disjoint union of open sets since $B_r(x)$ is clopen. Since this is true for all $r > 0$, $\{x\}$ is the largest connected component containing x.

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From these elementary results we already see that p-adic analysis and complex analysis are very different. As a further example, it follows from (g) that there are no arcs or paths from $[0, 1]$ to $\mathbb{K}$, or in fact to any ultrametric space.

Theorem (Combined Stone-Weierstrass theorem)

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{Q}_p, \mathbb{C}_p\}$ and let X be a non-empty compact subset of $\mathbb{K}$. Let $(A, \|\cdot\|_\infty)$ be a Banach $\mathbb{K}$-subalgebra of $C_{\mathbb{K}}(X)$ satisfying:

(i) *A includes each element of $\mathbb{K}$ as a constant function,*

(ii) *A separates the points of X,*

(iii) *And, if $\mathbb{K} = \mathbb{C}$, A is self adjoint i.e. $f \in A \Leftrightarrow \bar{f} \in A$,*

Then $A = C_{\mathbb{K}}(X)$.

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Hence, for $\mathbb{K} = \mathbb{Q}_p$ or $\mathbb{C}_p$, $C_{\mathbb{K}}(X)$ has no nontrivial proper subalgebras, as in the case with $\mathbb{K} = \mathbb{R}$.