Generalising Uniform Algebras Over Complete Valued Fields

Jonathan Mason

School of Mathematical Sciences
University of Nottingham

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We begin with the following definition.

**Definition 1.0**

Let $F$ be a complete valued field.
Let $A$ be a commutative unital Banach $F$-algebra.
We say that $A$ has **finite basic dimension** if there exists a finite extension $L$ of $F$ extending $F$ as a valued field such that:

(i) for each proper closed prime ideal $J$ of $A$, that is the kernel of a bounded multiplicative seminorm on $A$, $\text{Frac}(A/J)$ is $F$-isomorphic to a subfield of $L$;

(ii) there is $g \in \text{Gal}(L/F)$ with $L^g = F$, where $L^g := \{x \in L : g(x) = x\}$. 


Representation of uniform algebras, overview.

Let $F$ be $\mathbb{C}$

let $A$ be a commutative unital Banach $F$-algebra with $\|a^2\| = \|a\|^2$ for all $a \in A$ and finite basic dimension.

$A$ is a complex uniform algebra on a compact Hausdorff space,
Representation of uniform algebras, overview.

Let $F$ be $\mathbb{C}$ or $\mathbb{R}$

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$\|a^2\| = \|a\|^2$ for all $a \in A$ and finite basic dimension.

$A$ is a complex uniform algebra

or $A$ is a real function algebra

on a compact Hausdorff space,

Hausdorff space,
Representation of uniform algebras, overview.

Let $F$ be $\mathbb{C}$ or $\mathbb{R}$ or $\mathbb{K}$, a locally compact complete nonarchimedean field, let $A$ be a commutative unital Banach $F$-algebra with $\|a^2\| = \|a\|^2$ for all $a \in A$ and finite basic dimension.

- $A$ is a complex uniform algebra on a compact Hausdorff space,
- or $A$ is a real function algebra on a compact Hausdorff space,
- or $A$ is a nonarchimedean analog of the real function algebras on a Stone space.

Note, here a Stone space is a totally disconnected compact Hausdorff space.
Definition 1.1

Let $F$ and $L$ be complete valued fields such that $L$ is an extension of $F$ as a valued field. Let $X$ be a compact Hausdorff space and let $C_L(X)$ be the Banach algebra of all continuous $L$-valued functions on $X$ with pointwise operations and the sup norm. If a subset $A$ of $C_L(X)$ satisfies:

(i) $A$ is closed under pointwise operations;

(ii) $A$ is complete with respect to $\| \cdot \|_\infty$;

(iii) $F \subset A$;

(iv) $A$ separates the points of $X$,

then we will call $A$ an $L/F$ uniform algebra or just a uniform algebra when convenient.
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In the language of Definition 1.1, an $L/F$ uniform algebra is a Banach $F$-algebra of $L$-valued functions.
We now generalise two definitions by Kulkarni and Limaye from the theory of real function algebras.
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**Definition 1.2 (J. Mason 2009)**

Let $F$ and $L$ be complete valued fields such that $L$ is a finite extension of $F$ as a valued field. Let $X$ be a compact Hausdorff space and totally disconnected if $F$ is nonarchimedean. Define,

$$C(X, \tau, g) := \{ f \in C_L(X) : f(\tau(x)) = g(f(x)) \text{ for all } x \in X \}$$

where:

(i) $g \in \text{Gal}(L/F)$;

(ii) $\tau : X \to X$ with $\text{ord}(\tau) | \text{ord}(g)$;

(iii) $g$ and $\tau$ are continuous.

We will call $C(X, \tau, g)$ the **basic $L/L^g$ function algebra** on $(X, \tau, g)$, where $L^g := \{ x \in L : g(x) = x \}$, or just a **basic function algebra** when convenient.
Definition 1.3 (J. Mason 2009)

Let $F$ and $L$ be complete valued fields such that $L$ is a finite extension of $F$ as a valued field. Let $(X, \tau, g)$ conform to the conditions of Definition 1.2 and let $A$ be a subset of the basic $L/L_g$ function algebra on $(X, \tau, g)$.

If $A$ is also an $L/L_g$ uniform algebra then we will call $A$ an $L/L_g$ function algebra on $(X, \tau, g)$. 
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Theorem 1.4

With respect to the above definitions the basic $L/L_g$ function algebra on $(X, \tau, g)$ is always an $L/L_g$ uniform algebra.
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Theorem 1.4

With respect to the above definitions the basic $L/L^g$ function algebra on $(X, \tau, g)$ is always an $L/L^g$ uniform algebra.

Note, in fact $\text{ord}(\tau)|\text{ord}(g)$ is an optimum condition in Definition 1.2 since if we do not include it in Definition 1.2 then $C(X, \tau, g)$ separates the points of $X$ if and only if $\text{ord}(\tau)|\text{ord}(g)$. 
Archimedean examples.
Archimedean examples.

(Ex1) Let $F = \mathbb{R}$, $L = \mathbb{C}$ and $X$ be a compact Hausdorff space. We have $\text{Gal}(\mathbb{C}/\mathbb{R}) = \{ \text{id}, \bar{z} \}$.

Setting $g = \text{id}$ forces $\tau$ to be the identity on $X$. In this case $C(X, \tau, g) = C_{\mathbb{C}}(X)$ and each $L^/_{L^g}$ function algebra on $(X, \tau, g)$ is a complex uniform algebra.
Archimedean examples.

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Setting $g = \text{id}$ forces $\tau$ to be the identity on $X$. In this case $C(X, \tau, g) = C_\mathbb{C}(X)$ and each $L/L_s$ function algebra on $(X, \tau, g)$ is a complex uniform algebra.

On the other hand, setting $g = \bar{z}$ forces $\tau$ to be a topological involution on $X$. In this case the $L/L_s$ function algebras on $(X, \tau, g)$ are precisely the real function algebras of Kulkarni and Limaye.
Nonarchimedean examples.
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(Ex2) Let $F = \mathbb{Q}_5$, $L = \mathbb{Q}_5(\sqrt{2})$ with the unique extension of the 5-adic valuation and $X := \{x \in L : |x|_L \leq 1\}$.

Let $g$ be the Galois automorphism that sends $\sqrt{2}$ to $-\sqrt{2}$. Here $g$ is an isometry on $L$ and so we can take $\tau = g$. In this case $C(X, \tau, g)$ has the property that every power series in $C(X, \tau, g)$ has $\mathbb{Q}_5$ valued coefficients. However, since $X \subset \mathbb{Q}_5(\sqrt{2})$ these power series are $\mathbb{Q}_5(\sqrt{2})$ valued functions.
(Ex3) Let $F, L, X$ and $g$ be as in Ex2.

We can obtain a function $\omega : L \to \mathbb{Z} \cup \{+\infty\}$ such that for all $x \in L$ we have $|x|_L = 5^{-\omega(x)}$.

Define $\tau(0) = 0$ and for $x \in X \setminus \{0\}$,

$$
\tau(x) := \begin{cases} 
5x & \text{if } 2 \mid \omega(x) \\
5^{-1}x & \text{if } 2 \nmid \omega(x).
\end{cases}
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In this case the only power series in $C(X, \tau, g)$ are constants belonging to $\mathbb{Q}_5$. 
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However there are elements of $C(X, \tau, g)$ that when restricted to a circle in $X$ about the origin, can be expressed as a power series on the circle.
Before introducing the next theorem we recall the definition below.

**Definition 1.0**

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(i) for each proper closed prime ideal $J$ of $A$, that is the kernel of a bounded multiplicative seminorm on $A$, Frac$(A/J)$ is $F$-isomorphic to a subfield of $L$;

(ii) there is $g \in \text{Gal}(L/F)$ with $L^g = F$, where $L^g := \{x \in L : g(x) = x\}$. 
We have the following representation result.
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**Theorem 1.5 (J. Mason 2010)**

Let $F$ be a locally compact complete nonarchimedean valued field with nontrivial valuation.
Let $A$ be a commutative unital Banach $F$-algebra with $\|a^2\| = \|a\|^2$ for all $a \in A$ and finite basic dimension.
Then:

(i) for some finite extension $L$ of $F$ extending $F$ as a valued field, a character space $\mathcal{M}(A)$ of $L$ valued, multiplicative $F$-linear functionals can be defined;

(ii) the space $\mathcal{M}(A)$ is a totally disconnected compact Hausdorff space;

(iii) $A$ is isometrically $F$-isomorphic to a $L/F$ function algebra on $(\mathcal{M}(A), g, g)$ for some $g \in \text{Gal}(L/F)$. 
Residue algebra theorem.
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**Theorem 1.6 (J. Mason 2010)**

Let $F$ be a locally compact complete nonarchimedean valued field of characteristic zero with nontrivial valuation. Let $L$ be a finite unramified extension of $F$ with $L^g = F$ for some $g \in \text{Gal}(L/F)$ and let $C(X, \tau, g)$ be a basic $L/F$ function algebra. Then:

(i) $\mathcal{O} := \{f \in C(X, \tau, g) : \|f\|_\infty \leq 1\}$ is a ring;

(ii) $\mathcal{J} := \{f \in C(X, \tau, g) : \|f\|_\infty < 1\}$ is an ideal of $\mathcal{O}$;

(iii) $\mathcal{O}/\mathcal{J} \cong C(X, \tau, \bar{g})$ where $C(X, \tau, \bar{g})$ is the basic $\bar{L}/\bar{F}$ function algebra on $(X, \tau, \bar{g})$. Here $\bar{F}$ and $\bar{L}$ are respectively the residue fields of $F$ and $L$ whilst $\bar{g}$ is the residue automorphism on $\bar{L}$ induced by $g$. 
(Q1) An open question. Wermer gave the following theorem in 1963.

**Theorem**

Let $X$ be a compact Hausdorff space, $A \subseteq \mathbb{C}(X)$ a complex uniform algebra and $\text{Re}A := \{ \text{Re} f : f = \text{Re}f + i\text{Im}f \in A \}$. If $\text{Re}A$ is a ring then $A = \mathbb{C}(X)$.

An analog of Wermer’s theorem for real function algebras was given by Kulkarni and Srinivasan in 1990.

**Theorem**

Let $X$ be a compact Hausdorff space, $\tau$ a topological involution on $X$ and $A$ a $\mathbb{C}/\mathbb{R}$ function algebra on $(X, \tau, \bar{z})$. If $\text{Re}A$ is a ring then $A = \mathbb{C}(X, \tau, \bar{z})$.

Can this be generalised further for $\mathbb{C}(X, \tau, g)$?
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**Theorem**

Let $X$ be a compact Hausdorff space, $A \subseteq C(X)$ a complex uniform algebra and $\text{Re}A := \{\text{Re} f = \text{Re} f + i\text{Im} f \in A\}$. If $\text{Re}A$ is a ring then $A = C(X)$. 

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Can this be generalised further for $C(X, \tau, g)$?
(Q2) For $f \in C_L(X)$ define $\sigma(f) := g^{\text{ord}(g)-1} \circ f \circ \tau$.
We have $f \in C(X, \tau, g)$ if and only if $\sigma(f) = f$.
Does every higher order algebraic involution on $C_L(X)$ has the form $\sigma$ for some $g$ and $\tau$?
Aside, for $g$ an isometry on $L$ we automatically have that $\sigma$ is an isometry on $C_L(X)$. 
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There are many open questions.
References:

(1) V. G. Berkovich, Spectral theory and analytic geometry over nonarchimedean fields, Mathematical surveys and monographs, no. 33, American Mathematical Society, 1990.


We will end here.
We begin with some elementary p-adic analysis to reveal some striking differences with complex analysis.
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**Definition**

Let $K$ be a field. A **nonarchimedean** absolute value on $K$ is a function $|·| : K \rightarrow \mathbb{R}$ such that for any $a, b \in K$ we have:

1. $|a| \geq 0$ with $|a| = 0$ if and only if $a = 0$,
2. $|ab| = |a| \cdot |b|$,
3. $|a + b| \leq \max\{|a|, |b|\}$, strong triangle inequality.

If $K$ is complete with respect to the metric obtained from $|·|$ then $K$ is called nonarchimedean.

More generally, a metric space $(X, d)$ is called an ultrametric space if the metric $d$ satisfies the strong triangle inequality, $d(x, z) \leq \max\{d(x, y), d(y, z)\}$. 
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More generally, a metric space $(X, d)$ is called an **ultrametric space** if the metric $d$ satisfies the strong triangle inequality,

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}.$$
For each prime \( p \in \mathbb{N} \) there is a nonarchimedean absolute value \(| \cdot |_p\) on the field of rational numbers \( \mathbb{Q} \). The completion of \( \mathbb{Q} \) obtained using \(| \cdot |_p\) is the nonarchimedean field \( \mathbb{Q}_p \) of p-adic numbers.
For each prime $p \in \mathbb{N}$ there is a nonarchimedean absolute value $| \cdot |_p$ on the field of rational numbers $\mathbb{Q}$. The completion of $\mathbb{Q}$ obtained using $| \cdot |_p$ is the nonarchimedean field $\mathbb{Q}_p$ of $p$-adic numbers.

Each $x \in \mathbb{Q}_p^\times$ has a unique $p$-power series expansion of the form

$$x = \sum_{i \leq n} a_n p^n, \quad a_n \in \{0, \cdots, p - 1\}, \quad a_i \neq 0, \quad i \in \mathbb{Z}.$$ 

The absolute value of $x \neq 0$ is then $|x|_p = p^{-i}$ and $|0|_p = 0$. 
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The completion of the algebraic closure of $\mathbb{Q}_p$ is denoted $\mathbb{C}_p$ and is fortunately algebraically closed. $\mathbb{C}_p$ is a nonarchimedean field extending $\mathbb{Q}_p$. 
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The completion of the algebraic closure of $\mathbb{Q}_p$ is denoted $\mathbb{C}_p$ and is fortunately algebraically closed. $\mathbb{C}_p$ is a nonarchimedean field extending $\mathbb{Q}_p$.

$\mathbb{Q}_p$ is locally compact where as $\mathbb{C}_p$ is not. Further $\mathbb{C}_p$ and $\mathbb{C}$ are isomorphic, $\mathbb{C}_p \cong \mathbb{C}$ as fields.
**p-adic balls** in $\mathbb{K} := \mathbb{Q}_p$ or $\mathbb{C}_p$.

Let $r \in \mathbb{R}$, $r > 0$ and let $\leq$ be one of $<$ or $\leq$ on $\mathbb{R}$. 

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**Overview Definitions Examples Theorems Open Questions References Extra**
**p-adic balls** in \( \mathbb{K} := \mathbb{Q}_p \) or \( \mathbb{C}_p \).

Let \( r \in \mathbb{R}, r > 0 \) and let \( \leq \) be one of \(<\) or \(\leq\) on \( \mathbb{R} \).

Then \( x \sim y :\iff |x - y|_p \leq r \) is an equivalence relation on \( \mathbb{K} \) by the strong triangle inequality. To show transitivity, let \( x \sim y \) and \( y \sim z \) then,

\[
|x - z|_p = |x - y + y - z|_p \leq \max\{|x - y|_p, |y - z|_p\} \leq r \quad \text{so} \quad x \sim z.
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$$

**(a)** Hence every $\mathbb{K}$ ball $B_r(x)$ is an equivalence class and so every point in $B_r(x)$ is at it’s center because every element is an equivalence class representative. Hence every $\mathbb{K}$ ball is open.
**p-adic balls** in $\mathbb{K} := \mathbb{Q}_p$ or $\mathbb{C}_p$.

Let $r \in \mathbb{R}, r > 0$ and let $\leq$ be one of $<$ or $\leq$ on $\mathbb{R}$.

Then $x \sim y :\Leftrightarrow |x - y|_p \leq r$ is an equivalence relation on $\mathbb{K}$ by the strong triangle inequality. To show transitivity, let $x \sim y$ and $y \sim z$ then,

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(a) Hence every $\mathbb{K}$ ball $B_r(x)$ is an equivalence class and so every point in $B_r(x)$ is at it’s center because every element is an equivalence class representative. Hence every $\mathbb{K}$ ball is open.

(b) Algebraically, $\mathbb{K}/_\sim := \{B_r(x) : x \in \mathbb{K}\}$ is an Abelian group.
**p-adic balls** in \( K := \mathbb{Q}_p \) or \( \mathbb{C}_p \).

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(a) Hence every \( K \) ball \( B_r(x) \) is an equivalence class and so every point in \( B_r(x) \) is at it’s center because every element is an equivalence class representative. Hence every \( K \) ball is open.

(b) Algebraically, \( K/\sim := \{B_r(x) : x \in K\} \) is an Abelian group.

(c) Since \( K \) is a disjoint union of \( \sim \) equivalence classes, \( K \) is a disjoint union of balls of radius \( r \).
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Let $r \in \mathbb{R}$, $r > 0$ and let $\leq$ be one of $<$ or $\leq$ on $\mathbb{R}$.

Then $x \sim y :\Leftrightarrow |x - y|_p \leq r$ is an equivalence relation on $\mathbb{K}$ by the strong triangle inequality. To show transitivity, let $x \sim y$ and $y \sim z$ then,

$$|x - z|_p = |x - y + y - z|_p \leq \max\{|x - y|_p, |y - z|_p\} \leq r$$

so $x \sim z$.

(a) Hence every $\mathbb{K}$ ball $B_r(x)$ is an equivalence class and so every point in $B_r(x)$ is at its center because every element is an equivalence class representative. Hence every $\mathbb{K}$ ball is open.

(b) Algebraically, $\mathbb{K}/\sim := \{B_r(x) : x \in \mathbb{K}\}$ is an Abelian group.

(c) Since $\mathbb{K}$ is a disjoint union of $\sim$ equivalence classes, $\mathbb{K}$ is a disjoint union of balls of radius $r$.

(d) It follows easily that for $y \notin B_r(x)$ we have $B_r(y) \cap B_r(x) = \emptyset$. Hence, also noting (a), every $\mathbb{K}$ ball is clopen.
(e) Also from (c) for any two balls $B_{r_1}(x)$ and $B_{r_2}(y)$, either they are disjoint or one is a subset of the other.
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(g) $\mathbb{K}$ is totally disconnected.
   To see this note that for all $r > 0$ and for all $x \in \mathbb{K}$, $\mathbb{K} = (\mathbb{K} \setminus B_r(x)) \cup B_r(x)$ is a disjoint union of open sets since $B_r(x)$ is clopen. Since this is true for all $r > 0$, $\{x\}$ is the largest connected component containing $x$. 
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From these elementary results we already see that $p$-adic analysis and complex analysis are very different. As a further example, it follows from (g) that there are no arcs or paths from $[0, 1]$ to $\mathbb{K}$, or in fact to any ultrametric space.
Theorem (Combined Stone-Weierstrass theorem)

Let $K \in \{\mathbb{R}, \mathbb{C}, \mathbb{Q}_p, \mathbb{C}_p\}$ and let $X$ be a non-empty compact subset of $K$. Let $(A, \| \cdot \|_\infty)$ be a Banach $K$-subalgebra of $C_K(X)$ satisfying:

(i) $A$ includes each element of $K$ as a constant function,
(ii) $A$ separates the points of $X$,
(iii) And, if $K = \mathbb{C}$, $A$ is self adjoint i.e. $f \in A \iff \overline{f} \in A$,

Then $A = C_K(X)$. 
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Hence, for $K = \mathbb{Q}_p$ or $\mathbb{C}_p$, $C_K(X)$ has no nontrivial proper subalgebras, as in the case with $K = \mathbb{R}$. 